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A variational principle and conservation theorems in connexion with the generally relativistic Dirac equation*

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Introduction

In a recent paper Klein [1] has proposed a treatment of the gravitational field based on the use of the γ and Γ matrices from the theory of the generally covariant Dirac equation (see Bargmann [2]). Klein introduces an action integral containing these matrices and the two sets are varied independently. This procedure bears a certain analogy to the Palatini method, in which the components of the metric tensor and of the affine connexion are varied independently. The Palatini method has proved to be well suited for attacking the problem of quantizing the gravitational field (Arnowitt and Deser [3]). One result obtained here is that the equations one obtains with Klein's variation principle are the Einstein equations if the variations of the γ matrices are restricted so that

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$$

where the $g^{\mu\nu}$ are multiples of the unit matrix, and are also the components of the tensor reciprocal to the metric tensor (see section I.4.).

In section I the method with independent variations of the γ and Γ matrices is extended to a gravitational field interacting with a Dirac field.[1] (This corresponds to Weyl's [4] extension of the Palatini method.) This is done in section I.5, where the new equations are obtained and compared with the Einstein and Dirac equations. The sections preceding this one are preparative in nature. Sections I.1–3 constitute a summary of the Bargmann paper [2] with slight generalizations and section I.4 is devoted to a comparative review of the usual variation principle with coupled variations of the γ and the Γ matrices.

In section II we derive an "energy-momentum complex" from the above-mentioned Lagrangian without interaction. This is done in section II.1 following the procedure used by Møller in [5] (see also Laurent [6]); the resulting "complex" is very similar to Møller's. In section II.2 the two are compared. Section II.3 deals briefly with a conservation identity connected with the S transformation.

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[1] For simplicity the Dirac field for a zero-mass particle is chosen.

I. Field equations

I.1. *Notation*

In the generally relativistic theory of the Dirac equation one introduces the four 4×4 matrices γ^μ ($\mu = 0, 1, 2, 3$), functions of the space-time coordinates, which together form a contravariant vector field. The anticommutators of these matrices are required to be multiples of the unit matrix,[1]

$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} \cdot 1, \tag{1}$$

and the tensor field $g^{\mu\nu}$ is identified with the reciprocal of the metric field $g_{\mu\nu}$. Indices are raised and lowered with the help of $g^{\mu\nu}$ and $g_{\mu\nu}$.

It is convenient to use the following definitions:

$$s^{\mu\nu} \equiv \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]; \tag{2}$$

$$\gamma_5 \equiv \frac{1}{24}\, \varepsilon^{\alpha\beta\gamma\delta}\, \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta; \tag{3}$$

$$g \equiv -\operatorname{Det}(g_{\mu\nu}), \tag{4}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, antisymmetric in all indices and with $\varepsilon^{0123} = 1$. Under coordinate transformations $s^{\mu\nu}$ is a contravariant tensor, γ_5 is a scalar density of weight 1, and g is a scalar density of weight 2. Equation (1) is covariant also under an arbitrary "S transformation",

$$(\gamma^\mu)^t = S^{-1} \gamma^\mu S, \tag{5}$$

where the matrix S generally is a function of the coordinates.

All the quantities defined above transform in this manner under S transformations. They are tensor-operators in Bargmann's [2] terminology. In Appendix 1 a few relations between these quantities are collected.

The covariant derivative defined with help of the $g_{\mu\nu}$ will, as usual, be denoted by a semicolon (e.g. $\gamma^\mu{}_{;\nu}$). Thus we have

$$g_{\alpha\beta;\gamma} = 0. \tag{6}$$

The $\gamma^\mu{}_{;\nu}$ and similar quantities derived from other tensor-operators are not generally covariant under S transformations. We can, however, with help of a new matrix field, Γ_μ, construct a quantity

$$\gamma^\mu{}_{|\alpha} \equiv \gamma^\mu{}_{;\alpha} + [\gamma^\mu, \Gamma_\alpha] \tag{7}$$

(with corresponding equations for other tensor-operators) which is a new tensor-operator. For this purpose we let Γ_α transform as a covariant vector under coordinate transformations and in the following manner under S transformations:

$$(\Gamma_\alpha)^t = S^{-1} \Gamma_\alpha S - S^{-1} S_{,\alpha}, \tag{8}$$

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[1] In the following, unit matrices will generally not be written out explicitly.

where the comma denotes the ordinary partial derivative. It is easy to check, with the help of (8), that $\gamma^{\mu}{}_{|\alpha}$ is, in fact, a tensor-operator and that the ordinary rule for differentiating a product applies also to the "stroke-operation" on a product of tensor-operators. We will give the operation (7) the name "extended covariant derivative" (e.c.d.).

A new tensor-operator

$$\Phi_{\mu\nu} \equiv \Gamma_{\nu,\mu} - \Gamma_{\mu,\nu} - [\Gamma_{\mu}, \Gamma_{\nu}] \tag{9}$$

can be constructed from a Γ field. It is easy to check directly, with the help of (8), that $\Phi_{\mu\nu}$ is, in fact, a tensor-operator.

Finally, we notice that although Γ_{μ} is not a tensor-operator, its variation, $\delta\Gamma_{\mu}$, is. This can be used to express the variation of $\Phi_{\mu\nu}$ in the following convenient way[1]

$$\delta\Phi_{\mu\nu} = (\delta\Gamma_{\nu})_{|\mu} - (\delta\Gamma_{\mu})_{|\nu}, \tag{10}$$

which follows immediately from varying (9).

I.2. *The Γ' field*

In the following we shall use different Γ_{μ} fields, all characterized by being covariant vector fields and transforming according to (8) under S transformations.

Any Γ_{μ} field can be used for constructing an e.c.d. but, of course, one obtains a different e.c.d. for each Γ_{μ} field. We shall distinguish between them by putting primes on the e.c.d. stroke, e.g.

$$\gamma^{\mu}{}_{|'\nu} \equiv \gamma^{\mu}{}_{;\nu} + [\gamma^{\mu}, \Gamma'_{\nu}]. \tag{11}$$

The field denoted by Γ'_{ν} is defined through the equation

$$\gamma^{\mu}{}_{|'\nu} = 0. \tag{12}$$

This equation can be solved for the Γ'_{ν} by expanding it in the η matrices of Appendix 1. The coefficients in this expansion may then be determined with the help of the relations (A 1.9–12). The result is

$$\begin{aligned}\Gamma'_{\nu} = -\tfrac{1}{8}\{\tfrac{1}{3}\operatorname{Tr}(s_{\alpha\varrho}\gamma^{\sigma}{}_{;\nu})\gamma^{\alpha} + \tfrac{1}{2}\operatorname{Tr}(\gamma_{\beta}\gamma_{\alpha;\nu})s^{\alpha\beta} + g^{-1}\operatorname{Tr}(\gamma_5\gamma_{\alpha;\nu})\gamma_5\gamma^{\alpha} - \\ - \tfrac{1}{4}g^{-1}\operatorname{Tr}(\gamma_5\gamma_{\varrho}\gamma^{\varrho}{}_{;\nu})\gamma_5\} + a_{\nu}.\end{aligned} \tag{13}$$

The quantity a_{ν} is here an arbitrary vector field transforming in the following manner under an S transformation

$$(a_{\mu})^{t} = a_{\mu} - \tfrac{1}{4}|S|^{-1}|S|_{,\mu}. \tag{14}$$

One can always, with the help of an S transformation, make[2]

$$\gamma^{\mu}{}_{,\nu} = C^{\mu}{}_{\nu\alpha}\gamma^{\alpha}, \tag{15}$$

where the $C^{\mu}{}_{\nu\alpha}$ are multiples of the unit matrix. It is often convenient to use this special system; then (13), for example, becomes

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[1] This corresponds to Palatini's formula giving the variation of the curvature tensor in terms of the variation of the affinity (see e.g. Møller [7] p. 334).

[2] cf. Bargmann's [2] "distinguished" (ausgezeichnet) system.

$$\Gamma'_\nu = \frac{1}{16} \operatorname{Tr} (\gamma_\alpha \gamma_{\beta;\nu}) s^{\alpha\beta} + a_\nu \tag{16}$$

if use is made of (A 1. 8). Notice that $\operatorname{Tr}(\gamma_\alpha \gamma_{\beta;\nu})$ is antisymmetric in α and β due to (1) and (6).

A field $\Phi'_{\mu\nu}$ is obtained from Γ'_ν through the general formula (9). It is easy to show, with the help of a locally inertial system and (15) (hence (16) may be used), that[1]

$$\Phi'_{\mu\nu} = \tfrac{1}{4} R_{\varkappa\lambda\nu\mu}\, s^{\varkappa\lambda} + a_{\nu,\mu} - a_{\mu,\nu}, \tag{17}$$

where $R_{\varkappa\lambda\nu\mu}$ is the curvature tensor. It is immediately seen that this equation is covariant in all respects and so we can drop our restrictions.

Using (A 1.8) and (A 1.10) we obtain from (17)

$$R_{\varrho\sigma\mu\nu} = \tfrac{1}{2} \operatorname{Tr} (s_{\varrho\sigma} \Phi'_{\mu\nu}). \tag{18}$$

The contracted form of this equation appears in Klein [1].

I.3. *Dirac fields*

In Bargmann's [2] formalism the Dirac fields are scalars transforming in the following manner under S transformations

$$\psi^t = S^{-1} \psi. \tag{19}$$

We shall, with Bargmann, introduce a quantity α such that

$$-\gamma^{\mu +} = \alpha \gamma^\mu \alpha^{-1} \tag{20}$$

and

$$\alpha^+ - \alpha = 0, \tag{21}$$

where the cross stands for hermitian conjugate. The matrix α is determined from (20) and (21) to within a real factor. The matrix α is taken to be a scalar, transforming under S transformations as follows

$$\alpha^t = S^+ \alpha S. \tag{22}$$

One can easily check that (22) makes (20) and (21) covariant. A tensor with the property (22) is, in Bargmann's terminology, a tensor-α, while a tensor with the property (19) is a tensor-ψ.

We shall quite generally require from the e.c.d. that the ordinary rule for differentiation of a product applies and that it and hermitian conjugation are commuting operations. With the following definition for the e.c.d. of a tensor-α

$$\alpha_{|\mu} = \alpha_{;\mu} + \alpha \Gamma_\mu + \Gamma_\mu^+ \alpha \tag{23}$$

the two rules check with (20) and (21) and $\alpha_{|\mu}$ is a new tensor-α.

The relation

$$\alpha_{|'\varrho} = 0 \tag{24}$$

does not contradict anything so far said about α. It gives a connexion between the undetermined fields in Γ_μ and α. In the following we shall always assume (24) to be valid.

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[1] Write out the covariant derivative in (16) explicitly

$$\gamma_{\beta;\nu} = \gamma_{\beta,\nu} - \Gamma^{\varrho}{}_{\beta\nu} \gamma_\varrho .$$

If the e.c.d. of a tensor-ψ is defined as follows

$$\psi_{|\mu} \equiv \psi_{;\mu} - \Gamma_\mu \psi \tag{25}$$

the two rules still apply, e.g. for products of the type $\psi^+ \alpha \gamma^\mu \psi$ (which are tensor-operators).

Using the two rules and (23), (24) and (25) we can further deduce the expression for the e.c.d. of the quantity $\bar{\psi} \equiv \psi^+ \alpha$:

$$\bar{\psi}_{|\mu} \equiv \bar{\psi}_{;\mu} + \bar{\psi}\, \Gamma_\mu. \tag{26}$$

It is often convenient to use the quantities

$$\varphi \equiv g^{\frac{1}{4}} \psi \tag{27}$$

and

$$\bar{\varphi} \equiv \varphi^+ \alpha \tag{28}$$

as substitutions for ψ and $\bar{\psi}$ (Klein [8]).

I.4. *The action containing* Γ'_μ.

The action for the gravitational field coupled to a massless Dirac field is the following

$$\int (\Re + 2\varkappa\, \bar{\varphi} \gamma^\mu \varphi_{|'\mu})\, d^4x, \tag{29}$$

where

$$\Re = g^{\frac{1}{2}} R$$

and R is the curvature scalar: $R = R^{\alpha\beta}{}_{\beta\alpha}$.

To obtain the field equations from (29) one must vary $g_{\mu\nu}$, φ and $\bar{\varphi}$ independently and set the corresponding variations of (29) equal to zero. Now, instead of varying the $g_{\mu\nu}$, we can vary the γ^μ in such a way that $\{\gamma^\mu, \gamma^\nu\}$ is always a multiple of the unit matrix, i.e.

$$\delta \gamma^\mu = \epsilon^\mu{}_\nu \cdot \gamma^\nu, \tag{30}$$

where $\epsilon^\mu{}_\nu$ is an infinitesimal mixed tensor field. The symmetric part of $\epsilon^\mu{}_\nu$ corresponds to a variation of $g^{\mu\nu}$, $\epsilon^{\mu\nu} + \epsilon^{\nu\mu} = \delta g^{\mu\nu}$, while the antisymmetric part corresponds to a class of S transformations and consequently gives no new field equations, since (29) is invariant under all S transformations. Hence we will in the following assume $\epsilon^{\mu\nu}$ to be symmetric.

Using (18) and performing a partial integration we can replace (29) by

$$\int \{ \mathrm{Tr}\,(\mathfrak{s}^{\alpha\beta} \Phi'_{\alpha\beta}) - 2\varkappa\, (\bar{\varphi} \gamma^\alpha \varphi_{|'\alpha} - \bar{\varphi}_{|'\alpha} \gamma^\alpha \varphi) \}\, d^4x. \tag{31}$$

If we introduce the explicit expressions for $\varphi_{|'\mu}$ and $\bar{\varphi}_{|'\mu}$,

$$\varphi_{|'\mu} = \varphi_{,\mu} - \tfrac{1}{2} \Gamma^\alpha{}_{\alpha\mu} \varphi - \Gamma'_\mu \varphi, \tag{32}$$

$$\bar{\varphi}_{|'\mu} = \bar{\varphi}_{,\mu} - \tfrac{1}{2} \Gamma^\alpha{}_{\alpha\mu} \bar{\varphi} + \bar{\varphi}\, \Gamma'_\mu, \tag{33}$$

into (31) we see that the middle terms in (32) and (33) cancel (Klein [8]). Consequently we need not consider them when performing the variations.

The variation δa_ν in (A 2.4) is not determined by $\delta \gamma^\mu$, but we see that if we do not wish to obtain a vanishing current

$$\bar{\varphi}\gamma^{\mu}\varphi=0$$

as a part of the field equations, we must restrict the variation of Γ'_{μ} so that $\delta a_{\nu}=0$, i.e.

$$\operatorname{Tr}(\delta\Gamma'_{\mu})=0. \tag{34}$$

Equation (34) will be used throughout.

The field equations are now obtained in a fairly straightforward manner. The variations of $\bar{\varphi}$ and φ yield

$$\gamma^{\mu}\varphi_{|'\mu}=0 \tag{35}$$

and

$$\bar{\varphi}_{|'\mu}\gamma^{\mu}=0. \tag{36}$$

In the *first* term of (31) the variation of $\Phi'_{\alpha\beta}$ gives no contribution due to (10) and (12), while the variation of $\mathfrak{g}^{\alpha\beta}$ give us the following term in the variation of (31)

$$-4\int \in^{\mu}{}_{\alpha}(\mathfrak{R}^{\alpha}{}_{\mu}-\tfrac{1}{2}\delta^{\alpha}_{\mu}\mathfrak{R})\,d^4x.$$

Equations (A 2.2) and (18) have been used here.

From the *second* term of (31) we obtain the following term by varying γ^{μ}

$$-2\varkappa\int[\bar{\varphi}\,\delta\gamma^{\mu}\varphi_{|'\mu}-\bar{\varphi}_{|'\mu}\delta\gamma^{\mu}\varphi-\bar{\varphi}\{\gamma^{\mu},\delta\Gamma'_{\mu}\}\varphi]\,d^4x.$$

We notice here that the last term vanishes according to (A 2.4) and (A 1.22). The use of (30) gives the final form to our expression, i.e.

$$-\varkappa\int[\in^{\mu\alpha}(\bar{\varphi}\gamma_{\alpha}\varphi_{|'\mu}-\bar{\varphi}_{|'\mu}\gamma_{\alpha}\varphi+\bar{\varphi}\gamma_{\mu}\varphi_{|'\alpha}-\bar{\varphi}_{|'\alpha}\gamma_{\mu}\varphi)]\,d^4x$$

and setting the variation of the total action (31) equal to zero gives us the Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\varkappa T_{\mu\nu}$$

with

$$T_{\mu\nu}=\tfrac{1}{4}(\bar{\psi}\gamma_{\mu}\psi_{|'\nu}-\bar{\psi}_{|'\nu}\gamma_{\mu}\psi+\bar{\psi}\gamma_{\nu}\psi_{|'\mu}-\bar{\psi}_{|'\mu}\gamma_{\nu}\psi). \tag{37}$$

1.5 The action with the γ and Γ fields not coupled

In this section we shall use the action (cf. (31))

$$\int\{\operatorname{Tr}(\mathfrak{g}^{\alpha\beta}\Phi_{\alpha\beta})-2\varkappa(\bar{\varphi}\gamma^{\alpha}\varphi_{|\alpha}-\bar{\varphi}_{|\alpha}\gamma^{\alpha}\varphi)\}\,d^4x \tag{38}$$

which contains derivatives up to the first order only of the quantities φ, $\bar{\varphi}$, γ^{μ} and Γ_{μ}.[1] These will be varied independently, γ^{μ} according to (30) and Γ_{μ} under the restriction (34), i. e.[2]

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[1] The first part of this action was proposed by Klein in [1].

[2] One could fear to lose, by this restriction, some equations obtained from variations of Γ_{μ} corresponding to S transformations and needed for the antisymmetric part of the "$\in^{\mu}{}_{\nu}$-equa-

$$\operatorname{Tr}(\delta\Gamma_\mu)=0. \tag{39}$$

We notice that the remark made in connexion with (32) and (33) is still valid.

a. Variation of Γ_μ.

Using (10) we obtain from (38) when varying Γ_μ

$$\int [2\operatorname{Tr}(\delta\Gamma_\mu \mathfrak{s}^{\mu\nu}{}_{|\nu}) + 2\varkappa\bar{\varphi}\{\gamma^\alpha, \delta\Gamma_\alpha\}\varphi]\,d^4x=0.$$

Considering the restriction (39) we arrive at the following equations for Γ_μ

$$\mathfrak{s}^{\mu\nu}{}_{|\nu} = -\varkappa\{\gamma^\mu, \varphi\bar{\varphi}\} + \frac{\varkappa}{2}\bar{\varphi}\gamma^\mu\varphi \tag{40}$$

($\varphi\,\bar{\varphi}$ is a square matrix).

We wish to solve (40) for Γ_μ. To that end we write

$$\Gamma_\mu = \Gamma'_\mu + M_\mu, \tag{41}$$

where M_μ is a tensor-operator from which we require that it fulfills

$$\operatorname{Tr}(M_\mu)=0. \tag{42}$$

From (40) and (41) it follows that

$$[s^{\mu\nu}, M_\nu] = -\varkappa\{\gamma^\mu, \psi\bar{\psi}\} + \frac{\varkappa}{2}\bar{\psi}\gamma^\mu\psi. \tag{43}$$

M_ν can be solved from equations (42) and (43) by expanding it in the η matrices of Appendix 1 and determining the coefficients. The result is

$$M_\mu = \frac{\varkappa}{4}\left[\tfrac{1}{8}\bar{\psi}\{s_{\mu\alpha}, \gamma_\beta\}\psi\, s^{\alpha\beta} + g^{-1}\bar{\psi}\gamma_5 s_{\mu\alpha}\psi\gamma_5\gamma^\alpha - \tfrac{1}{3}\bar{\psi}\psi\gamma_\mu\right] + c_\mu\gamma_5, \tag{44}$$

where the fact was used, that a three-index quantity symmetric in one pair and antisymmetric in another pair of indices must vanish. The c-number, weight -1, vector density, c_μ, represents the only arbitrariness of the solution. It is convenient to give expression (44) without the last term a special symbol

$$m_\mu \equiv M_\mu - c_\mu\gamma_5. \tag{45}$$

It is seen by inspection of (44) that

$$[\gamma^\mu, m_\mu]=0. \tag{46}$$

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tions" (see eq. (30)) to vanish automatically. However, the only "S variation" (see (A 2.10)) equation lost by the restriction (39) is easily seen to be

$$(\bar{\varphi}\gamma^\alpha\varphi)_{,\alpha}=0$$

and this equation is obtained also from the variation of φ and $\bar{\varphi}$ (see (57)).

This in turn gives us the relation

$$\gamma^{\mu}{}_{|\mu} = -2\, c_{\mu}\, \gamma_5\, \gamma^{\mu}. \tag{47}$$

We could now determine $\Phi_{\mu\nu}$ from (9), but this is rather complicated and not necessary for what follows. Hence we shall be content with finding the expression for $\operatorname{Tr}(s^{\varrho\sigma}\,\Phi_{\sigma\nu})$. Appendix 3 shows that

$$\operatorname{Tr}(s_{\varrho}{}^{\sigma}\,\Phi_{\sigma\nu}) = 2\,R_{\varrho\nu} + \frac{\varkappa}{4}\,(\bar{\psi}\,\{s_{\nu}{}^{\sigma},\gamma_{\varrho}\}\,\psi)_{;\sigma} + \varkappa\,\bar{\psi}\,\{m_{\nu},\gamma_{\varrho}\}\,\psi. \tag{48}$$

b. Variation of φ and $\bar{\varphi}$

From the action (38), when $\bar{\varphi}$ and φ are varied, it follows without calculation that

$$\gamma^{\mu}\,\varphi_{|\mu} + \tfrac{1}{2}\,\gamma^{\mu}{}_{|\mu}\,\varphi = 0 \tag{49}$$

and

$$\bar{\varphi}_{|\mu}\,\gamma^{\mu} + \tfrac{1}{2}\,\bar{\varphi}\,\gamma^{\mu}{}_{|\mu} = 0. \tag{50}$$

Introducing the new Γ field

$$\Gamma''_{\mu} \equiv \Gamma_{\mu} - c_{\mu}\,\gamma_5 \tag{51}$$

we see, with the help of (47), that (49) and (50) read

$$\gamma^{\mu}\,\varphi_{|''\mu} = 0 \tag{52}$$

and

$$\bar{\varphi}_{|''\mu}\,\gamma^{\mu} = 0, \tag{53}$$

in which equations the arbitrary field c_{μ} is canceled.

Equations (52) and (53) can also be written, using the Γ'_{μ} field, as

$$\gamma^{\mu}\,\psi_{|'\mu} - M\,\psi = 0 \tag{54}$$

and

$$\bar{\psi}_{|'\mu}\,\gamma^{\mu} + \bar{\psi}\,M = 0, \tag{55}$$

where

$$M \equiv \tfrac{1}{2}\{\gamma^{\mu}, M_{\mu}\} = \tfrac{1}{2}\{\gamma^{\mu}, m_{\mu}\} = \gamma^{\mu}\,m_{\mu} = m_{\mu}\,\gamma^{\mu}. \tag{56}$$

Equations (45) and (46) show the validity of the relations appearing in (56). Equations (54) and (55) show how the new Dirac equations differ from the ordinary ones, in which the last terms containing the matrix M are not found. In Appendix 4 a relatively simple expression for M is given. Equations (54) and (55) imply

$$(\bar{\varphi}\,\gamma^{\mu}\,\varphi)_{,\mu} = 0 \tag{57}$$

that is, the charge continuity equation.

c. Variation of γ^{μ}

Varying γ^{μ} and $\mathfrak{s}^{\mu\nu}$ in (38) according to (A2.1) and (A2.2) gives

$$\operatorname{Tr}(\Phi_{\mu\nu}\,s_{\alpha}{}^{\nu} - \tfrac{1}{2}\,g_{\alpha\mu}\,\Phi_{\varrho\sigma}\,s^{\varrho\sigma}) - \varkappa\,(\bar{\psi}\,\gamma_{\alpha}\,\psi_{|\mu} - \bar{\psi}_{|\mu}\,\gamma_{\alpha}\,\psi) + (\overset{\curvearrowright}{\alpha,\,\underset{\curvearrowleft}{\mu}}) = 0.$$

If we wish to compare these equations with Einstein's equations it is convenient to change the e.c.d.-s with respect to Γ_μ to e.c.d.-s with respect to Γ''_μ. This can be done without changing anything else in the equations. Furthermore we can use (48) in the first two terms noticing that the second term on the right-hand side is antisymmetric. The result is

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=$$

$$=-\frac{\varkappa}{4}(\bar\psi\gamma_\nu\psi_{|''\mu}-\bar\psi_{|''\mu}\gamma_\nu\psi-\bar\psi_{|''\nu}\gamma_\mu\psi+\bar\psi\gamma_\mu\psi_{|''\nu})-$$

$$-\frac{\varkappa}{2}\bar\psi\{\gamma_\mu, m_\nu\}\psi+\frac{\varkappa}{4}g_{\mu\nu}\bar\psi\{m_{\alpha,}\gamma^\alpha\}\psi.$$

In this expression the second term on the right-hand side cancels exactly the contribution of m_ν to the first term (notice that the second term is symmetric according to (A 3.3)), so that the e.c.d.-s with respect to Γ''_μ appearing there are changed to e.c.d.-s with respect to Γ'_μ. We obtain finally

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\varkappa(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\bar\psi M\psi), \tag{58}$$

where $T_{\mu\nu}$ and M are the quantities defined in (37) and (56) respectively.

Equation (58) shows how the "new Einstein's equations" differ from the ordinary ones, in which the term containing M does not appear. The new equations, (58), can be given a remarkably simple form if one notices that according to (37) and (54)–(55)

$$T^\alpha{}_\alpha=\bar\psi M\psi. \tag{59}$$

With the help of (59) the equations (58) obviously reduce to

$$R_{\mu\nu}=-\varkappa T_{\mu\nu}. \tag{60}$$

Hence, when $\varkappa=0$, the action principle employed here gives exactly Einstein's equations in vacuum.

II. Conservation theorems

In [5], Møller has shown that his well-known conservation theorems for energy and momentum in the theory of general relativity (see Møller [9]), in the form where $-\varkappa T_{\mu\nu}$ has been replaced by the left-hand side of Einstein's equations, are the identities derived from local varitions of the $g_{\mu\nu}$ in the curvature scalar.

In this section we shall treat the gravitational part of the Lagrangian in (31), i.e.

$$\mathrm{Tr}\,(\mathfrak{s}^{\alpha\beta}\Phi'_{\alpha\beta}), \tag{61}$$

in a corresponding manner, by varying γ^μ and Γ_μ locally (cf. Laurent [6]). The local variations of a few quantities are collected in Appendix 2. There and henceforth the local variation is denoted by the symbol δ.

In this section no other Γ field than the Γ' field will be used. For simplicity *we therefore omit the prime connected with Γ'_μ everywhere*, i.e.

$$\gamma^{\mu}{}_{|\nu}=0. \tag{62}$$

II. 1. *Conservation theorems connected with coordinate transformations.*

We shall here investigate how (61) changes under a local variation connected with the infinitesimal coordinate transformation

$$x'^{\mu}=x^{\mu}+\xi^{\mu}(x) \tag{63}$$

i.e. a variation of type (a) in Appendix 2.

Using (10) and (62) followed by (A 2.6) and a few partial integrations we obtain

$$\delta\int \mathrm{Tr}\,(\mathfrak{s}^{\mu\nu}\,\Phi_{\mu\nu})\,d^4x=$$

$$=-\int \mathrm{Tr}\,\{(\Phi_{\sigma\nu}\,\mathfrak{s}^{\varrho\nu})_{,\varrho}+(\Phi_{\mu\sigma}\,\mathfrak{s}^{\mu\varrho})_{,\varrho}-(\Phi_{\mu\nu}\,\mathfrak{s}^{\mu\nu})_{,\sigma}+\Phi_{\mu\nu}\,\mathfrak{s}^{\mu\nu}{}_{,\sigma}\}\,\xi^{\sigma}d^4x.$$

This variation of (61) vanishes, however, due to the invariance of (61) under coordinate transformations and we obtain the identity

$$\mathrm{Tr}\,\{(\Phi_{\sigma\nu}\,\mathfrak{s}^{\varrho\nu})_{,\varrho}-\tfrac{1}{2}\,\Phi_{\mu\nu,\sigma}\,\mathfrak{s}^{\mu\nu}\}\equiv 0. \tag{64}$$

This identity is closely related to the identity

$$(\mathfrak{R}^{\alpha\beta}-\tfrac{1}{2}g^{\alpha\beta}\,\mathfrak{R})_{;\beta}\equiv 0,$$

which is obtained in a corresponding manner from the action $\int \mathfrak{R}\,d^4x=0$. Furthermore

$$\delta\,\mathrm{Tr}\,(\Phi_{\mu\nu}\,\mathfrak{s}^{\mu\nu})=-\mathrm{Tr}\,(\Phi_{\mu\nu}\,\mathfrak{s}^{\mu\nu}\,\xi^{\varrho})_{,\varrho} \tag{65}$$

due to the fact that (61) is a scalar density. On the other hand is

$$\delta\,\mathrm{Tr}\,(\Phi_{\mu\nu}\,\mathfrak{s}^{\mu\nu})=$$

$$=-2\,\mathrm{Tr}\,\{(\Gamma_{\varrho}\,\xi^{\varrho}{}_{,\nu}+\Gamma_{\nu,\varrho}\,\xi^{\varrho})\,\mathfrak{s}^{\mu\nu}\}_{,\mu}+\mathrm{Tr}\,\{\Phi_{\mu\nu}\,(\mathfrak{s}^{\sigma\nu}\,\xi^{\mu}{}_{,\sigma}+\mathfrak{s}^{\mu\sigma}\,\xi^{\nu}{}_{,\sigma}-\mathfrak{s}^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma}-\mathfrak{s}^{\mu\nu}{}_{,\sigma}\,\xi^{\sigma})\}, \tag{66}$$

where (10), (A 2.7) and (A 2.6) have been used.

Identifying the coefficients for ξ^{ϱ} in the right-hand sides of (65) and (66) gives the following relation

$$\mathrm{Tr}\,(\Phi_{\varrho\nu}\,\mathfrak{s}^{\sigma\nu}-\Gamma_{\nu,\varrho}\,\mathfrak{s}^{\sigma\nu})_{,\sigma}\equiv 0, \tag{67}$$

where we have made use of (64).

Identifying the coefficients for $\xi^{\varrho}{}_{,\sigma}$ gives

$$\mathrm{Tr}\,(\Phi_{\varrho\nu}\,\mathfrak{s}^{\sigma\nu}-\Gamma_{\nu,\varrho}\,\mathfrak{s}^{\sigma\nu})\equiv \mathrm{Tr}\,(\Gamma_{\varrho}\,\mathfrak{s}^{\mu\sigma})_{,\mu}, \tag{68}$$

while the identification of the coefficients of $\xi^{\varrho}{}_{,\sigma\tau}$ gives nothing.

With help of the antisymmetric superpotential

$$\Lambda_{\alpha}{}^{\beta\gamma} = -\mathrm{Tr}\,(\Gamma_{\alpha}\,\mathfrak{S}^{\beta\gamma}) \tag{69}$$

we can thus according to (68) and (67) define a divergence-free quantity

$$\mathfrak{T}_{\alpha}{}^{\beta} = \Lambda_{\alpha}{}^{\beta\gamma}{}_{,\gamma} \tag{70}$$

$$\mathfrak{T}_{\alpha}{}^{\beta}{}_{,\beta} = 0. \tag{71}$$

In the following section we shall discuss the relation of this quantity to the Møller energy-momentum complex.

II. 2. *Discussion of the new superpotential*

The superpotential $\Lambda_{\alpha}{}^{\beta\gamma}$ in (69) may be written in a different form

$$\begin{aligned} \Lambda_{\alpha}{}^{\beta\gamma} &= -\tfrac{1}{2}\sqrt{g}\,\mathrm{Tr}\,([\gamma^{\gamma}, \Gamma_{\alpha}]\,\gamma^{\beta}) = \\ &= \tfrac{1}{2}\sqrt{g}\,\mathrm{Tr}\,(\gamma^{\gamma}{}_{;\alpha}\,\gamma^{\beta}) = \\ &= \tfrac{1}{2}\sqrt{g}\,\mathrm{Tr}\,(\gamma^{\gamma}{}_{,\alpha}\,\gamma^{\beta}) + 2\,\Gamma^{\gamma}{}_{\sigma\alpha}\,\mathfrak{g}^{\sigma\beta}. \end{aligned}$$

The superpotential $\Lambda_{\alpha}{}^{\beta\gamma}$ is, however, antisymmetric in β and γ. It may thus be written

$$\Lambda_{\alpha}{}^{\beta\gamma} = \chi_{\alpha}{}^{\beta\gamma} - \tfrac{1}{4}\sqrt{g}\,\mathrm{Tr}\,(\gamma^{\beta}{}_{,\alpha}\,\gamma^{\gamma} - \gamma^{\gamma}{}_{,\alpha}\,\gamma^{\beta}), \tag{72}$$

where

$$\chi_{\alpha}{}^{\beta\gamma} \equiv -\Gamma^{\beta}{}_{\sigma\alpha}\,\mathfrak{g}^{\sigma\gamma} + \Gamma^{\gamma}{}_{\sigma\alpha}\,\mathfrak{g}^{\sigma\beta} \tag{73}$$

is Møller's [9] superpotential.

Our derivation of $\mathfrak{T}_{\alpha}{}^{\beta}$ is purely formal in nature and we have, of course, no right to call it *the* energy-momentum complex. One circumstance that casts doubt on such an interpretation is that neither $\mathfrak{T}_{\alpha}{}^{\beta}$ nor $\Lambda_{\alpha}{}^{\beta\gamma}$ is invariant under S transformations.

It is tempting to guess that the last term in (72) can be completely transformed away by an S transformation, in which case $\Lambda_{\alpha}{}^{\beta\gamma}$ would be "almost" identical with Møller's superpotential. To investigate if this is the case seems, however, to be a rather complicated undertaking. Let us instead briefly discuss the properties of $\Lambda_{\alpha}{}^{\beta\gamma}$ which can be seen immediately.

From (69) it is clear that $\Lambda_{\alpha}{}^{\beta\gamma}$, in distinction to $\chi_{\alpha}{}^{\beta\gamma}$, is a tensor density and from (72) it follows that $\Lambda_{0}{}^{\beta\gamma}$ is invariant under time-independent S transformations. It is also clear from the same formula that $\Lambda_{0}{}^{\beta\gamma}$ goes over into Møller's $\chi_{0}{}^{\beta\gamma}$ in the case of a static γ field. This in turn has as a consequence that $\mathfrak{T}_{0}{}^{0}$ integrated over the space of a closed system gives the correct result.

It is also interesting to note that the essential property of the Møller $\mathfrak{T}_{0}{}^{\alpha}$, that it transforms as a vector density under purely spatial transformations, is a property of $\Lambda_{0}{}^{\alpha\beta}{}_{,\beta}$ also. Equation (72) shows this immediately.

It is perhaps not necessary to point out that $\Lambda_{\alpha}{}^{\beta\gamma}{}_{,\gamma}$ transforms as a tensor density under *linear* transformations.

II. 3. *Conservation theorem connected with the S transformation*

Exactly as we have used the invariance of (61) under coordinate transformations, we may use its invariance under S transformations to derive identities in the form of divergence relations. The S transformation invariance

$$\delta \int \mathrm{Tr}\,(\Phi_{\mu\nu}\,\mathfrak{S}^{\mu\nu})\,d^4x = 0$$

gives, with the use of (10) and (A 2.9),

$$[\Phi_{\mu\nu}, \mathfrak{S}^{\mu\nu}] \equiv 0. \tag{74}$$

It further follows from

$$\delta\,\mathrm{Tr}\,(\Phi_{\mu\nu}\,\mathfrak{S}^{\mu\nu}) = 0$$

that, following the same pattern of derivation as in section II.1, but using formulae (10), (A2.9), (A2.10) and (74), we obtain

$$[\mathfrak{S}^{\mu\nu}, \Gamma_\nu]_{,\mu} \equiv 0 \tag{75}$$

$$[\mathfrak{S}^{\mu\nu}, \Gamma_\nu] \equiv -\mathfrak{S}^{\mu\nu}{}_{,\nu}\,. \tag{76}$$

Conclusion

The purpose of the first section of this paper was to investigate the changes in the Dirac and Einstein equations when one breaks the relation (12) between the γ and Γ matrices in the combined action (31) for the gravitation and Dirac fields, and varies the two sets of matrices independently. The result are contained in formulae (54), (55) and (58). It should be pointed out that these equations differ from the Dirac and Einstein equations only by small terms of the order $\varkappa^2$ and any experimental verification of one set or the other is at present far beyond reach. Now when it is clear that the variational principle leads to a reasonable result, the main interest would probably lie not so much in the second order equations as in the action itself, which may be used subsequently as the starting point for a quantization procedure.

In the second section we showed the simplicity of the procedure for obtaining conservation identities when one starts with the gravitational part of the action (31). In particular the superpotentials obtained are automatically antisymmetric. We also saw there that the new superpotentials are closely related to those introduced by Møller.

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APPENDIX 1

We introduce the following tensor-operators

$$\eta_1 \equiv 1, \tag{A 1.1}$$

$$\eta_2{}^{\mu} \equiv \gamma^{\mu}, \tag{A 1.2}$$

$$\eta_3{}^{\mu\nu} \equiv s^{\mu\nu}, \tag{A 1.3}$$

$$\eta_4{}^{\mu} \equiv \gamma_5 \gamma^{\mu}, \tag{A 1.4}$$

$$\eta_5 \equiv \gamma_5. \tag{A 1.5}$$

In a coordinate system in which the metric at a certain point is the Lorentz metric, these quantities go over, at that point, into the five tensors used in the special relativistic theory of Fermi particles (see e.g. Jauch and Rohrlich [10] p. 425).

Using such a coordinate system it is easy to show the validity of the following covariant (also under S transformations) equations collected here for reference

$$\{\gamma_5, \gamma^{\mu}\} = 0, \tag{A 1.6}$$

$$(\gamma_5)^2 = -g, \tag{A 1.7}$$

and

$$\operatorname{Tr}(\eta_r \eta_s) = 0, \text{ if } r \neq s. \tag{A 1.8}$$

The tensor indices on the η_r are suppressed here.

Furthermore

$$\operatorname{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4 g^{\mu\nu}, \tag{A 1.9}$$

$$\operatorname{Tr}(s^{\alpha\beta} s^{\mu\nu}) = 4(g^{\alpha\nu} g^{\beta\mu} - g^{\alpha\mu} g^{\beta\nu}), \tag{A 1.10}$$

$$\operatorname{Tr}(\gamma_5 \gamma^{\mu} \gamma_5 \gamma^{\nu}) = 4 g g^{\mu\nu}, \tag{A 1.11}$$

$$\operatorname{Tr}(\gamma_5 \gamma_5) = -4g, \tag{A 1.12}$$

and

$$[s^{\alpha\beta}, s^{\mu\nu}] = 2(g^{\beta\mu} s^{\alpha\nu} + g^{\alpha\mu} s^{\nu\beta} + g^{\nu\beta} s^{\mu\alpha} + g^{\nu\alpha} s^{\beta\mu}), \tag{A 1.13}$$

$$[\gamma_5 \gamma^{\mu}, \gamma_5 \gamma^{\nu}] = 2 g s^{\mu\nu}, \tag{A 1.14}$$

$$[s^{\mu\nu}, \gamma^{\alpha}] = 2(g^{\nu\alpha}\gamma^{\mu} - g^{\mu\alpha}\gamma^{\nu}), \tag{A 1.15}$$

$$[\gamma_5 \gamma^{\mu}, \gamma^{\nu}] = 2\gamma_5 g^{\mu\nu}, \tag{A 1.16}$$

$$[\gamma_5, \gamma^{\mu}] = 2\gamma_5 \gamma^{\mu}, \tag{A 1.17}$$

$$[\gamma_5 \gamma^{\alpha}, s^{\mu\nu}] = 2\gamma_5 (g^{\mu\alpha}\gamma^{\nu} - g^{\nu\alpha}\gamma^{\mu}), \tag{A 1.18}$$

$$[\gamma_5, s^{\mu\nu}] = 0, \tag{A 1.19}$$

$$[\gamma_5 \gamma^{\mu}, \gamma_5] = 2 g \gamma^{\mu}, \tag{A 1.20}$$

and

$$\{s^{\alpha\beta}, \gamma^{\gamma}\} = \{s^{\beta\gamma}, \gamma^{\alpha}\} = \{s^{\gamma\alpha}, \gamma^{\beta}\}, \tag{A 1.21}$$

$$\{s^{\alpha\beta},\gamma^{\gamma}\}=-\{s^{\alpha\gamma},\gamma^{\beta}\}=-\{s^{\gamma\beta},\gamma^{\alpha}\}, \tag{A 1.22}$$

and

$$\gamma_5\, s^{\alpha\beta}=-\tfrac{1}{2}\,\varepsilon^{\alpha\beta\varrho\sigma}\,s_{\varrho\sigma}, \tag{A 1.23}$$

$$\{\gamma^{\alpha},s^{\beta\gamma}\}=-2g^{-1}\,\varepsilon^{\alpha\beta\gamma\sigma}\,\gamma_5\gamma_\sigma. \tag{A 1.24}$$

Finally we remark that any 4×4 matrix can be expanded in the sixteen η matrices.

APPENDIX 2

The following variation of γ^μ

$$\delta\gamma^\mu=\epsilon^{\mu}{}_{\nu}\gamma^\nu \tag{A 2.1}$$

gives, for the variation $\mathfrak{s}^{\mu\nu}\equiv g^{\frac{1}{2}}\,s^{\mu\nu}$,

$$\delta\,\mathfrak{s}^{\mu\nu}=\epsilon^{\mu}{}_{\alpha}\,\mathfrak{s}^{\alpha\nu}+\epsilon^{\nu}{}_{\alpha}\,\mathfrak{s}^{\mu\alpha}-\epsilon^{\alpha}{}_{\alpha}\,\mathfrak{s}^{\mu\nu}. \tag{A 2.2}$$

The corresponding variation of the affine connexion is, as is well known,

$$\delta\,\Gamma^{\alpha}{}_{\beta\gamma}=\tfrac{1}{2}g^{\alpha\varrho}\,[(\delta\,g_{\varrho\beta})_{;\gamma}+(\delta\,g_{\varrho\gamma})_{;\beta}-(\delta\,g_{\beta\gamma})_{;\varrho}]$$

or, introducing the $\epsilon^{\mu}{}_{\nu}$,

$$\delta\,\Gamma^{\alpha}{}_{\beta\gamma}=-\,g^{\alpha\varrho}\,[\epsilon_{\varrho\beta;\gamma}+\epsilon_{\varrho\gamma;\beta}-\epsilon_{\beta\gamma;\varrho}]. \tag{A 2.3}$$

Using this formula we obtain

$$\delta\,\Gamma'_{\nu}=\tfrac{1}{2}\,\epsilon_{\nu\varrho;\sigma}\,s^{\varrho\sigma}+\delta\,a_\nu. \tag{A 2.4}$$

Equation (A 2.4) follows, e.g., by comparing the expressions for $\delta\,(\gamma^{\mu}{}_{|'\nu})$ and $(\delta\,\gamma^\mu)_{|'\nu}$, the first of which equals zero. The quantity a_ν is the vector field appearing in (13).

Below we have collected local variations corresponding in (*a*) to the infinitesimal coordinate transformation $x'^{\mu}=x^\mu+\xi^\mu(x)$, and in (*b*) to the infinitesimal S transformation with $S=1+\sigma$.

(*a*)

$$\delta\,\gamma^\mu=\gamma^\sigma\,\xi^{\mu}{}_{,\sigma}-\gamma^{\mu}{}_{,\sigma}\,\xi^\sigma \tag{A 2.5}$$

$$\delta\,\mathfrak{s}^{\mu\nu}=\mathfrak{s}^{\sigma\nu}\,\xi^{\mu}{}_{,\sigma}+\mathfrak{s}^{\mu\sigma}\,\xi^{\nu}{}_{,\sigma}-\mathfrak{s}^{\mu\nu}\,\xi^{\sigma}{}_{,\sigma}-\mathfrak{s}^{\mu\nu}{}_{,\sigma}\,\xi^\sigma \tag{A 2.6}$$

$$\delta\,\Gamma_\mu=-\,\Gamma_\sigma\,\xi^{\sigma}{}_{,\mu}-\Gamma_{\mu,\sigma}\,\xi^\sigma \tag{A 2.7}$$

(*b*)

$$\delta\,\gamma^\mu=[\gamma^\mu,\sigma] \tag{A 2.8}$$

$$\delta\,\mathfrak{s}^{\mu\nu}=[\mathfrak{s}^{\mu\nu},\sigma] \tag{A 2.9}$$

$$\delta\,\Gamma_\mu=[\Gamma_\mu,\sigma]-\sigma_{,\,\mu} \tag{A 2.10}$$

APPENDIX 3

The simplest way of finding the expression for $\mathrm{Tr}\,(s^{\varrho\sigma}\Phi_{\sigma\nu})$ in (48) is to use a locally inertial system, in which the Γ'_μ and the first derivatives of the γ^μ vanish locally. Then

$$\mathrm{Tr}\,(s^{\varrho\sigma}\Phi_{\sigma\nu}) = \mathrm{Tr}\,(s^{\varrho\sigma}\Phi'_{\sigma\nu}) + \mathrm{Tr}\,(s^{\varrho\sigma}\{M_{\nu,\sigma} - M_{\sigma,\nu}\}) - \mathrm{Tr}\,(s^{\varrho\sigma}[M_\sigma, M_\nu]). \qquad (\mathrm{A}\,3.1)$$

The *first term* in this expression is known from (18).

The *second term* becomes, with the help of (44) and (A 1.10),

$$\frac{\varkappa}{4}\,(\bar\psi\,\{s_\nu{}^\sigma, \gamma^\varrho\}\,\psi)_{,\sigma}.$$

The *third term* may be changed in the following manner, using (43) and (42),

$$\begin{aligned}
&-\mathrm{Tr}\,(s^{\varrho\sigma}[M_\sigma, M_\nu]) = \\
&= -\mathrm{Tr}\,([s^{\varrho\sigma}, M_\sigma]\,M_\nu) = \\
&= \varkappa\,\mathrm{Tr}\,(\{\gamma^\varrho, \psi\,\bar\psi\}\,M_\nu) = \\
&= \varkappa\,\bar\psi\,\{M_\nu, \gamma^\varrho\}\,\psi.
\end{aligned}$$

Hence the complete expression reads

$$\mathrm{Tr}\,(s^{\varrho\sigma}\Phi_{\sigma\nu}) = 2\,R^\varrho{}_\nu + \frac{\varkappa}{4}\,(\bar\psi\,\{s_\nu{}^\sigma, \gamma^\varrho\}\,\psi)_{;\sigma} + \varkappa\,\bar\psi\,\{m_\nu, \gamma^\varrho\}\,\psi, \qquad (\mathrm{A}\,3.2)$$

where the comma in the middle term is changed to a semicolon so that (A 3.2) becomes covariant. Further M_ν has been changed to m_ν in the last term. We are allowed to make this change, according to (45) and (A 1.6).

We notice that the last term in (A 3.2) is symmetric in ν and ϱ, so that

$$\bar\psi\,\{m_\nu, \gamma_\mu\}\,\psi = \bar\psi\,\{m_\mu, \gamma_\nu\}\,\psi. \qquad (\mathrm{A}\,3.3)$$

This follows immediately from (44).

APPENDIX 4

In this appendix we shall simplify the explicit expression for M, defined in (56). Using (44) in (56) gives

$$M = \frac{\varkappa}{8}\,[\tfrac{1}{8}\,\bar\psi\,\{s_{\alpha\beta}, \gamma_\gamma\}\,\psi\,\{s^{\alpha\beta}, \gamma^\gamma\} - 2\,g^{-1}\,\bar\psi\,\gamma_5\,s_{\alpha\beta}\,\psi\,\gamma_5\,s^{\alpha\beta} - \tfrac{8}{3}\,\bar\psi\,\psi]. \qquad (\mathrm{A}\,4.1)$$

An alternative expression can be found using (A 1.23) and (A 1.24)

$$M=\frac{\varkappa}{8}[-3g^{-1}\bar{\psi}\gamma_5\gamma^{\varrho}\psi\gamma_5\gamma_{\varrho}+2\bar{\psi}s^{\varrho\sigma}\psi s_{\varrho\sigma}-\tfrac{8}{3}\bar{\psi}\psi]. \tag{A 4.2}$$

We have here used the facts that

$$\varepsilon_{\alpha\beta\gamma\varrho}\varepsilon^{\alpha\beta\gamma\sigma}=-6g\,\delta_{\varrho}^{\sigma}$$

and

$$\varepsilon_{\alpha\beta\varrho\sigma}\varepsilon^{\alpha\beta\varkappa\lambda}=-2g\,(\delta_{\varrho}^{\varkappa}\delta_{\sigma}^{\lambda}-\delta_{\sigma}^{\varkappa}\delta_{\varrho}^{\lambda}).$$

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